Due Sun

1.4 – Inverses; Algebraic Properties of Matrices

Theorem 1.4.1 Properties of Matrix Arithmetic

Assuming that the sizes of the matrices are such that the indicated operations can be performed, the following rules of matrix arithmetic are valid for matrices A, B, and C and scalars a, b, and c.

a) $A + B = B + A$
(commutative law for matrix addition)

b) $A + (B + C) = (A + B) + C = A + B + C$
(associative law for matrix addition)

c) $(AB)C = A(BC) = ABC$
(associative law for matrix multiplication)

d) $A(B + C) = AB + AC$ (left distributive law)

e) $(B + C)A = BA + CA$ (right distributive law)

f) $A(B - C) = AB - AC$

g) $(B - C)A = BA - CA$

h) $a(B + C) = aB + aC$

i) $a(B - C) = aB - aC$

j) $(a + b)C = aC + bC$

k) $(a - b)C = aC - bC$

l) $a(bC) = (ab)C$

m) $a(BC) = (aB)C = B(aC)$

Pf (h): By assumption, B & C are the same size. Scalar multiplication does not change the size of a matrix. Thus, $a(B+C)$, aB, and aC are all the same size.

$(a(B+C))_{ij}$ is the entry in the i^{th} row and j^{th} column of the matrix on the LHS.

$$(a(B+C))_{ij} = a(B+C)_{ij} = a(b_{ij} + c_{ij})$$

$$= a\,b_{ij} + a\,c_{ij}$$ distributive property of real #s.

$$= (aB)_{ij} + (aC)_{ij}$$

Since corresponding entries are equal,

$$a(B+C) = aB + aC.$$ ✓

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prove $a(B+C) = aB + aC$

$$a(B+C) = a\left(\begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & & & \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix} + \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & & & \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix}\right)$$

$$= a\begin{bmatrix} b_{11}+c_{11} & b_{12}+c_{12} & \cdots & b_{1n}+c_{1n} \\ b_{21}+c_{21} & b_{22}+c_{22} & \cdots & b_{2n}+c_{2n} \\ \vdots & & & \\ b_{m1}+c_{m1} & b_{m2}+c_{m2} & \cdots & b_{mn}+c_{mn} \end{bmatrix}$$

$$= \begin{bmatrix} a(b_{11}+c_{11}) & \cdots & a(b_{1n}+c_{1n}) \\ & \text{and so on} & \end{bmatrix}$$

$$= \begin{bmatrix} ab_{11}+ac_{11} & \cdots & ab_{1n}+ac_{1n} \\ & \text{stuff} & \end{bmatrix}$$

$$= a\begin{bmatrix} b_{11} & \cdots & b_{1n} \\ & & \end{bmatrix} + a\begin{bmatrix} c_{11} & \cdots & c_{1n} \\ & & \end{bmatrix}$$

$$= aB + aC$$

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side note: <u>representing multiplication</u>

└ side note: <u>notation</u> $A = [a_{ij}]$ are matrices

$(A)_{ij} = a_{ij}$ are entries

Let $A = [a_{ij}]$ be an $m \times k$ matrix and

$B = [b_{ij}]$ be a $k \times n$ matrix.

$$\qquad\qquad m \times \boxed{k \qquad\qquad k} \times n$$

$$AB = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ \vdots & & & \\ a_{i1} & a_{i2} & \cdots & a_{ik} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mk} \end{bmatrix} \begin{bmatrix} b_{11} & \cdots & b_{1j} & \cdots & b_{1n} \\ b_{21} & \cdots & b_{2j} & \cdots & b_{2n} \\ \vdots & & \vdots & & \vdots \\ b_{k1} & \cdots & b_{kj} & \cdots & b_{kn} \end{bmatrix}$$

$$(AB)_{ij} = a_{i\underline{1}}b_{\underline{1}j} + a_{i\underline{2}}b_{\underline{2}j} + \cdots + a_{i\underline{k}}b_{\underline{k}j}$$

We can write this as

$$(AB)_{ij} = \sum_{r=1}^{k} a_{ir} b_{rj}$$

Note:
$(AB)_{ij} \neq a_{ij} b_{ij}$

Note if $AC = BC$ it does not always follow that $A = B$.

In general, $AB \neq BA$. In the special cases where $AB = BA$, we say that A and B **commute**.

A **zero matrix**, denoted O, is a matrix whose entries are all zero.

Theorem 1.4.2 Properties of Zero Matrices
If c is a scalar, and if the sizes of the matrices A and O are such that the operations can be performed, then:

a) $A + O = O + A = A$
b) $A - O = A$
c) $A - A = A + (-A) = O$
d) $OA = O$
e) If $cA = O$, then $c = 0$ or $A = O$

We can have $A \neq O$ and $B \neq O$ but $AB = O$.

$$\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$$

Identity matrices are square matrices with 1's on the main diagonal and zeros everywhere else. They are denoted I or I_n if referencing the size, $n \times n$.

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad AI = IA = A$$

Theorem 1.4.3 If R is the reduced row echelon form of an $n \times n$ matrix A, then either R has at least one row of zeros or R is the identity matrix I_n.

Pf: An $n \times n$ matrix has at most n pivots. If there are n pivots in R, then $R = I_n$ by def of rref. If there are fewer than n pivots, then at least one row does not have a leading 1 and so is a row of zeros. ✓

Definition 1: If A is a square matrix, and if there exists a matrix B of the same size for which $AB = BA = I$, then A is said to be **invertible** (or **nonsingular**) and B is called the **inverse** of A. If no such matrix B exists, then A is said to be **singular**. Because the inverse of a matrix A is unique, we will denote it using A^{-1}.

Theorem 1.4.4 Uniqueness of a Matrix Inverse
If B and C are both inverses of the matrix A, then $B = C$.

$AC = I$ and $CA = I$

Pf:

associative prop of matrix mult.

$$B = B\underbrace{I}_{\text{property of the identity matrix}} = B\underbrace{(AC)}_{\text{C is an inverse of A}} = (BA)C \underbrace{=}_{\text{B is an inverse of A}} IC \underbrace{=}_{\text{property of I}} C$$

Theorem 1.4.5 The matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible if and only if $ad - bc \neq 0$, in which case the inverse is given by the formula $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

8. Use Theorem 1.4.5 to compute the inverse.

$\begin{bmatrix} 6 & 4 \\ -2 & -1 \end{bmatrix}$ $\quad ad - bc = 6(-1) - (-2)(4)$

$= -6 + 8 = 2$

$$A^{-1} = \frac{1}{2}\begin{bmatrix} -1 & -4 \\ 2 & 6 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & -2 \\ 1 & 3 \end{bmatrix}$$

Theorem 1.4.6 If A and B are invertible matrices with the same size, then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

Pf: Consider $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1}$

$= AIA^{-1}$

$= AA^{-1}$

$= I$

Since multiplying AB by $B^{-1}A^{-1}$ yields I,

$(AB)^{-1} = B^{-1}A^{-1}$ ✓

Theorem 1.4.7 Properties of Negative Exponents

If A is invertible and n is a nonnegative integer, then:

a) A^{-1} is invertible and $(A^{-1})^{-1} = A$.

b) A^n is invertible and $(A^n)^{-1} = A^{-n} = (A^{-1})^n$.

c) kA is invertible for any nonzero scalar k, and $(kA)^{-1} = k^{-1}A^{-1}$.

reciprocal

inverse

15. Use the given information to find A.

$(7A)^{-1} = \begin{bmatrix} -3 & 7 \\ 1 & -2 \end{bmatrix}$

23. Find all values of a, b, c, and d (if any) for which the matrices $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ commute.

AB commute $\Rightarrow AB = BA$

$$\Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & a \\ 0 & c \end{bmatrix} = \begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix}$$

$\boxed{\text{A \& B commute if } c=0 \text{ and } a=d. \text{ b can be any number.}}$

25. Solve the system using an inverse matrix.

$3x_1 - 2x_2 = -1$

$4x_1 + 5x_2 = 3$

$$A\vec{x} = \vec{b} \Rightarrow A^{-1}A\vec{x} = \underset{n \times n}{A^{-1}}\underset{n \times 1}{\vec{b}}$$

$$\vec{x} = A^{-1}\vec{b}$$

$$\begin{bmatrix} 3 & -2 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 4 & 5 \end{bmatrix}^{-1}\begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -2 \\ 4 & 5 \end{bmatrix}^{-1} = \frac{1}{15+8}\begin{bmatrix} 5 & 2 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} 5/23 & 2/23 \\ -4/23 & 3/23 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5/23 & 2/23 \\ -4/23 & 3/23 \end{bmatrix}\begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-5+6}{23} \\ \frac{4+9}{23} \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1/23 \\ 13/23 \end{bmatrix}$$

39. Simplify the expression assuming that A, B, C and D are invertible.

$(AB)^{-1}(AC^{-1})(D^{-1}C^{-1})^{-1}D^{-1}$

$B^{-1}A^{-1}AC^{-1}CDD^{-1} = B^{-1}$

Theorem 1.4.8 Properties of the Transpose

If the sizes of the matrices are such that the stated operations can be performed, then:

a) $(A^T)^T = A$

b) $(A + B)^T = A^T + B^T$

c) $(A - B)^T = A^T - B^T$

d) $(kA)^T = kA^T$

e) $(AB)^T = B^T A^T$

m×k k×n n×k k×m

Theorem 1.4.9 If A is an invertible matrix, then A^T is also invertible and $(A^T)^{-1} = (A^{-1})^T$.

Pf: A invertible $\Rightarrow A^{-1}$ exists

$A^T(A^{-1})^T = (A^{-1}A)^T = I^T = I$

Since multiplying A^T by $(A^{-1})^T$ yields I, $(A^T)^{-1} = (A^{-1})^T$

Revisiting $(AB)^T = B^T A^T$ (not a proof)